

Optimized maximum-confidence discrimination of N mixed quantum states and application to symmetric states

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We study an optimized measurement which discriminates N mixed quantum states occurring with given prior probabilities. The measurement yields the maximum achievable confidence for each of the N conclusive outcomes, thereby keeping the overall probability of inconclusive outcomes as small as possible. It corresponds to optimum unambiguous discrimination when for each outcome the confidence is equal to unity. Necessary and sufficient optimality conditions are derived and general properties of the optimum measurement are obtained. The results are applied to the optimized maximum-confidence discrimination of N equiprobable symmetric mixed states. Analytical solutions are presented for a number of examples, including the discrimination of N symmetric pure states spanning a d -dimensional Hilbert space ($d \leq N$) and the discrimination of N symmetric mixed qubit states.

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I. INTRODUCTION

The discrimination between different quantum states is an essential task in quantum communication, quantum cryptography and quantum computing. Since nonorthogonal states cannot be distinguished perfectly, various optimized discrimination strategies have been developed. The two best known of these are minimum-error discrimination [1] and optimum unambiguous discrimination [2–5]. In the latter strategy, errors are not allowed, at the expense of admitting inconclusive results the probability of which is minimized in the optimum measurement. Recently the optimum unambiguous discrimination between mixed states attracted a lot of interest [6–15]. Unambiguous discrimination of each single state in a given set of states is only possible when pure states are linearly independent [16], and when for mixed states the support [17] of each respective density operator is different from the support of every other density operator belonging to a state in the set [6, 7].

For the case when it is not possible to discriminate each single state unambiguously, Croke and co-workers [18, 19] introduced the strategy of maximum-confidence discrimination which was experimentally demonstrated for three symmetric pure qubit states [20]. In this strategy, we are as confident as possible that the respective state was indeed present when a conclusive result is obtained. Maximum-confidence discrimination corresponds to unambiguous discrimination when for each conclusive result the confidence is equal to unity. As with unambiguous discrimination, also with maximum-confidence discrimination in general the measurement is not unique and a further optimization can be performed which minimizes the number of inconclusive results, or, in other words, the failure probability of the discrimination measurement. The optimized maximum-confidence discrimination of two mixed qubit states has been theoretically investigated in our previous paper [21]. We also proposed an implementation of the optimum measurement

[22] which has been recently experimentally realized for two mixed qubit states having the same purity [23].

In the present paper we extend the theoretical investigations to the discrimination of an arbitrary number of mixed states. In Sec. II we describe the measurement for maximum-confidence discrimination of N mixed states. Section III is devoted to the problem of determining the optimum measurement which yields the smallest failure probability. Necessary and sufficient optimality conditions are derived and general properties of the optimum measurement are obtained. A proof concerning the necessity of the optimality conditions is given in the Appendix. In Sec. IV we apply our results to the discrimination of symmetric states and obtain analytical solutions for special cases. Section V concludes the paper. We add that recently a related publication appeared where maximum-confidence discrimination is treated for N symmetric pure qudit states [24].

II. DETECTION OPERATORS FOR MAXIMUM-CONFIDENCE DISCRIMINATION

We suppose that a quantum system is prepared with the prior probability η_j in one of N given states described by the density operators ρ_j ($j = 1, \dots, N$), where we assume that $\sum_{j=1}^N \eta_j = 1$. It will be convenient to introduce the density operator ρ characterizing the total information about the quantum system,

$$\rho = \sum_{j=1}^N \eta_j \rho_j = \sum_{l=1}^d r_l |r_l\rangle\langle r_l| \quad \text{with} \quad \sum_{l=1}^d |r_l\rangle\langle r_l| = I_d, \quad (2.1)$$

where for later use we introduced the spectral representation of ρ . The eigenstates $|r_l\rangle$ form a complete orthonormal basis in the d -dimensional Hilbert space \mathcal{H}_d jointly spanned by the eigenstates of ρ_1, \dots, ρ_N that belong to nonzero eigenvalues, and I_d is the identity operator in \mathcal{H}_d . We want to perform a measurement in order

to infer from a single outcome in which of the N possible states the system was prepared. The discrimination measurement is described by $N + 1$ positive detection operators $\Pi_0, \Pi_1, \dots, \Pi_N$ fulfilling the completeness relation $\sum_{j=0}^N \Pi_j = I_d$. The conditional probability that a system is inferred to be in the state ρ_j given it has been prepared in the state ρ_k reads $p(j|\rho_k) = \text{Tr}(\rho_k \Pi_j)$, while $\text{Tr}(\rho_k \Pi_0)$ is the conditional probability that the measurement yields an inconclusive result. From the completeness relation we get the requirement

$$\Pi_0 = I_d - \sum_{j=1}^N \Pi_j \geq 0, \quad \Pi_j \geq 0 \quad (j = 1, \dots, N). \quad (2.2)$$

The confidence in the conclusive measurement outcome j has been introduced as the conditional probability $p(\rho_j|j) = p(\rho_j, j)/p(j)$ that the state ρ_j was indeed prepared, given that the outcome j is detected [18]. Here $p(\rho_j, j) = \eta_j \text{Tr}(\rho_j \Pi_j)$ is the joint probability that the state ρ_j was prepared and the detector j clicks, and $p(j) = \text{Tr}(\rho \Pi_j)$ is the total probability for the detection of the outcome j . In other words, the confidence is the ratio between the number of instances when the outcome j is correct and the total number of instances when the outcome j is detected. In this paper we shall denote the maximum possible value of the confidence for the state j by C_j , that is,

$$C_j = \max \left\{ \frac{\eta_j \text{Tr}(\rho_j \Pi_j)}{\text{Tr}(\rho \Pi_j)} \right\} = \max \left\{ \text{Tr} \left[\tilde{\rho}_j \frac{\rho^{1/2} \Pi_j \rho^{1/2}}{\text{Tr}(\rho \Pi_j)} \right] \right\}, \quad (2.3)$$

where the maximum is taken with respect to all possible measurements. Here we have defined the transformed density operators

$$\tilde{\rho}_j = \rho^{-1/2} \eta_j \rho_j \rho^{-1/2} \quad \text{with} \quad \sum_{j=1}^N \tilde{\rho}_j = I_d. \quad (2.4)$$

The maximum confidence C_j is equal to the largest eigenvalue of the operator $\tilde{\rho}_j$, and it is obtained in a measurement where the operator $\rho^{1/2} \Pi_j \rho^{1/2}$ has its support in the eigenspace belonging to the largest eigenvalue of $\tilde{\rho}_j$ [18, 21]. Denoting the projector onto this eigenspace by P_j , we can write the spectral decomposition of $\tilde{\rho}_j$ as

$$\tilde{\rho}_j = C_j P_j + \sum_{k=m_j+1}^d \nu_k^{(j)} |\nu_k^{(j)}\rangle \langle \nu_k^{(j)}|, \quad (2.5)$$

where m_j is the degree of degeneracy of the largest eigenvalue and where

$$P_j = \sum_{k=1}^{m_j} |\nu_k^{(j)}\rangle \langle \nu_k^{(j)}|, \quad \sum_{k=1}^d |\nu_k^{(j)}\rangle \langle \nu_k^{(j)}| = I_d. \quad (2.6)$$

The detection operators describing a maximum-confidence measurement then can be represented as [21]

$$\Pi_j = \sum_{k,k'=1}^{m_j} a_{kk'}^{(j)} \rho^{-1/2} |\nu_k^{(j)}\rangle \langle \nu_{k'}^{(j)}| \rho^{-1/2} \quad (2.7)$$

($j = 1, \dots, N$) with suitably chosen coefficients $a_{kk'}^{(j)}$. Let us introduce the projector Λ_j onto the support of Π_j , that is

$$\Lambda_j = \text{projector onto span}\{\rho^{-1/2} |\nu_1^{(j)}\rangle, \dots, \rho^{-1/2} |\nu_{m_j}^{(j)}\rangle\}. \quad (2.8)$$

Equation (2.7) shows that a maximum-confidence measurement is defined by the property that

$$\Pi_j = \Lambda_j \Pi_j \Lambda_j \quad (j = 1, \dots, N) \quad (2.9)$$

which restricts the supports of the detection operators Π_j to the required subspaces. We can derive an explicit expression for the projectors Λ_j . From Eq. (2.8) we get $\Lambda_j \rho^{-1/2} |\nu_k^{(j)}\rangle = \rho^{-1/2} |\nu_k^{(j)}\rangle$ for $k = 1, \dots, m_j$ and thus $\Lambda_j \rho^{-1/2} P_j = \rho^{-1/2} P_j$ which yields the relation $\rho^{1/2} \Lambda_j \rho^{1/2} \rho^{-1} P_j = P_j$. The latter relation can be further modified. Taking into account that because of Eq. (2.8) the projector onto the support of $\rho^{1/2} \Lambda_j \rho^{1/2}$ is given by P_j we get $\rho^{1/2} \Lambda_j \rho^{1/2} P_j \rho^{-1} P_j = P_j$ and finally

$$\Lambda_j = \rho^{-1/2} (P_j \rho^{-1} P_j)^{-1} \rho^{-1/2}, \quad (2.10)$$

where we used the convention that the inverse of an operator is defined on its support, that is $(P_j \rho^{-1} P_j)^{-1} = P_j (P_j \rho^{-1} P_j)^{-1} P_j$. Clearly, $\Lambda_j^2 = \Lambda_j$, as expected for a projector. Since the ratios in Eq. (2.3) do not change when the operators Π_j ($j = 1, \dots, N$) are multiplied by arbitrary constants, it is always possible to construct a measurement where Eqs. (2.9) and (2.2) are fulfilled with a suitable detection operator Π_0 .

III. OPTIMIZED MEASUREMENT

A. Necessary and sufficient optimality conditions

In this paper we consider the measurement that discriminates N mixed quantum states with maximum confidence for each conclusive result, and that is optimized by the additional requirement that the overall failure probability Q be as small as possible. The latter is defined as the overall probability of inconclusive results, $Q = \text{Tr}(\rho \Pi_0)$. It is convenient to introduce the overall probability R that the measurement delivers a conclusive result, no matter whether this result is correct or wrong, given by

$$R = 1 - Q = \sum_{j=1}^N \text{Tr}(\rho \Pi_j) = \sum_{k,j=1}^N \eta_k \text{Tr}(\rho_k \Pi_j). \quad (3.1)$$

The task is now to determine the specific measurement where for each of the conclusive outcomes j the confidence takes its maximum possible value C_j while R is as large as possible. For this purpose we have to solve the

optimization problem

$$\text{maximize } R = \sum_{j=1}^N \text{Tr}(\Lambda_j \rho \Lambda_j \Pi_j), \text{ subject to } \Pi_0 \geq 0, \quad (3.2)$$

where Eq. (2.9) and the cyclic invariance of the trace has been used. This problem can be cast into a different form by observing that the relation

$$\text{Tr} Z - R = \text{Tr}(Z \Pi_0) + \sum_{j=1}^N \text{Tr}[\Lambda_j (Z - \rho) \Lambda_j \Pi_j] \quad (3.3)$$

is identically fulfilled for any operator Z , as can be seen after replacing Z on the left-hand side of Eq. (3.3) by $Z I_d$ with $I_d = \Pi_0 + \sum_j \Lambda_j \Pi_j \Lambda_j$. Since the detection operators are positive, we conclude from Eq. (3.3) that the inequality

$$\text{Tr} Z - R \geq 0 \quad (3.4)$$

holds true provided that the sufficient positivity conditions

$$Z \geq 0, \quad \Lambda_j (Z - \rho) \Lambda_j \geq 0 \quad (j = 1, \dots, N) \quad (3.5)$$

are satisfied. Equation (3.4) implies that the discrimination probability R cannot be larger than the smallest possible value of $\text{Tr} Z$. In other words, the minimum of $\text{Tr} Z$ under the constraints given by Eq. (3.5) determines an upper bound for R . In order to determine this bound, we thus arrive at the alternative optimization problem

$$\text{minimize } \text{Tr} Z, \text{ subject to Eq. (3.5).} \quad (3.6)$$

When the bound is reached, that is when $R = \min(\text{Tr} Z)$, the right-hand side of Eq. (3.3) vanishes for the optimum operator Z solving the minimization problem given in Eq. (3.6). Due to the positivity conditions in Eq. (3.5) and the positivity of the detection operators, all traces in Eq. (3.3) are taken over positive operators. Hence the expression on the right-hand side of Eq. (3.3) can only vanish when the conditions

$$Z \Pi_0 = 0, \quad \Lambda_j (Z - \rho) \Pi_j = 0 \quad (j = 1, \dots, N) \quad (3.7)$$

are fulfilled, where in the second equation again Eq. (2.9) has been used. Since the converse is obvious, Eq. (3.7) is necessary and sufficient for optimality, provided that the positivity conditions in Eq. (3.5) are fulfilled. As will be shown in the Appendix by a proof which is analogous to a recent proof concerning minimum-error discrimination [25], the conditions in Eq. (3.5) are not only sufficient, but also necessary for the validity of Eq. (3.4). Together Eqs. (3.5) and (3.7) then represent necessary and sufficient optimality conditions. When we can find an operator Z and positive detection operators Π_0 and $\Pi_j = \Lambda_j \Pi_j \Lambda_j$ ($j = 1, \dots, N$) which satisfy these conditions, then the detection operators determine the optimum measurement, that is the maximum-confidence

measurement which maximizes R , or minimizes the failure probability Q , respectively.

The results can be written in an alternative way by taking into account that

$$\Lambda_j \eta_j \rho_j \Lambda_j = C_j \Lambda_j \rho \Lambda_j \quad (3.8)$$

which follows from Eqs. (2.4), (2.5) and (2.10) using $P_j \tilde{\rho}_j P_j = C_j P_j \rho^{-1/2} \rho \rho^{-1/2} P_j$. We thus arrive at the necessary and sufficient optimality conditions

$$\Lambda_j (Z - \rho) \Lambda_j = \Lambda_j \left(Z - \frac{\eta_j \rho_j}{C_j} \right) \Lambda_j \geq 0, \quad Z \geq 0, \quad (3.9)$$

$$\Lambda_j (Z - \rho) \Pi_j = \Lambda_j \left(Z - \frac{\eta_j \rho_j}{C_j} \right) \Pi_j = 0, \quad Z \Pi_0 = 0, \quad (3.10)$$

with $\Pi_j = \Lambda_j \Pi_j \Lambda_j$ and $j = 1, \dots, N$. The rate of conclusive results reads

$$R = 1 - Q = \sum_{j=1}^N \text{Tr}(\rho \Pi_j) = \sum_{j=1}^N \frac{\eta_j}{C_j} \text{Tr}(\rho_j \Pi_j). \quad (3.11)$$

When $C_j = 1$ for $j = 1, \dots, N$ Eqs. (3.9) and (3.10) coincide with the conditions for optimum unambiguous discrimination that have been derived by Eldar *et al.* [8] from a semidefinite programming problem using duality theory in linear optimization. In this context, the optimization problem posed in Eq. (3.2) is called the primal problem, while the problem posed in Eq. (3.6) is denoted as the dual problem. The specialization to unambiguous discrimination will be discussed in more detail in Sec. III.D.

B. Rank of the optimum failure operator

From the optimality conditions we can draw a conclusion about the rank of the failure operator in the optimum measurement for maximum-confidence discrimination. Equation (3.7) implies that for $Z \neq 0$ the operator Π_0 is orthogonal to Z which means that the eigenstates of Π_0 cannot span the full d -dimensional Hilbert space \mathcal{H}_d introduced by means of Eq. (2.1). More precisely, the first equality in Eq. (3.7) requires that

$$\text{rank } Z + \text{rank } \Pi_0 \leq d \quad (3.12)$$

since otherwise the joint eigensystems of Z and Π_0 would consist of more than d states in \mathcal{H}_d and due to the linear dependence of these states the operators would not be orthogonal. On the other hand, the second positivity constraint in Eq. (3.5) implies that the relation $\text{rank}(\Lambda_j Z \Lambda_j) \geq \text{rank}(\Lambda_j \rho \Lambda_j)$ is fulfilled for each value of j . Since obviously $\text{rank } Z \geq \text{rank}(\Lambda_j Z \Lambda_j)$ we arrive at

$$\text{rank } Z \geq \max_j \{ \text{rank}(\Lambda_j \rho \Lambda_j) \} = \max_j \{ \text{rank}(\Lambda_j \rho_j \Lambda_j) \} \quad (3.13)$$

where for the last equality sign Eq. (3.8) has been taken into account. Combining Eqs. (3.12) and (3.13) we obtain the condition

$$\text{rank } \Pi_0 \leq d - \max_j \{\text{rank}(\Lambda_j \rho_j \Lambda_j)\} \quad (3.14)$$

which restricts the rank of the failure operator in a measurement performing maximum-confidence discrimination with minimum failure probability.

C. Dimensionality of the optimization problem

We introduce the d' -dimensional subspace $\mathcal{H}_{d'}$ jointly spanned by the projectors $\Lambda_1, \dots, \Lambda_N$. When we define

$$\rho' = \frac{\Lambda \rho \Lambda}{\text{Tr}(\rho \Lambda)}, \quad \Lambda = \text{projector onto span}(\Lambda_1, \dots, \Lambda_N), \quad (3.15)$$

the optimization problem posed in Eq. (3.2) corresponds to the maximization of

$$R = \text{Tr}(\rho \Lambda) R' \quad \text{with} \quad R' = \sum_{j=1}^N \text{Tr}(\rho' \Lambda_j \Pi_j \Lambda_j), \quad (3.16)$$

under the constraint that $\Pi_0 = I_d - \Lambda + \Pi'_0 \geq 0$. Here we introduced the operator $\Pi'_0 = \Lambda - \sum_{j=1}^N \Lambda_j \Pi_j \Lambda_j$ which has its support in $\mathcal{H}_{d'}$. Since $I_d - \Lambda$ is the projector onto the subspace which is orthogonal to $\mathcal{H}_{d'}$, the positivity constraint on Π_0 is satisfied provided that $\Pi'_0 \geq 0$. The optimization is thus reduced to the maximization of R' subject to $\Pi'_0 \geq 0$, which is a problem in a Hilbert space of dimension $d' \leq d$. The latter problem is formally equivalent to the original optimization problem when we identify Λ with the identity operator in $\mathcal{H}_{d'}$. Hence the condition

$$\text{span}(\Lambda_1, \dots, \Lambda_N) = \mathcal{H}_d, \quad \Lambda = I_d \quad (3.17)$$

defines a standard form of the optimization problem to which the general problem can be reduced with the help of Eqs. (3.15) and (3.16). It is therefore sufficient to restrict the investigations to the case $d' = d$. Eq. (3.16) shows that for $d' < d$ maximum-confidence discrimination without inconclusive results is never possible since in this case $\text{Tr}(\rho \Lambda) < 1$, which yields a rate of conclusive results $R < 1$ even when $R' = 1$, that is, even when $\Pi'_0 = 0$.

D. Specialization to unambiguous discrimination

When each of the given states can be discriminated with perfect confidence, that is for

$$C_1 = \dots = C_N = 1, \quad (3.18)$$

the maximum-confidence measurement is equivalent to a measurement which unambiguously discriminates between the N states. Indeed, from Eqs. (2.4) and (2.5)

we obtain the relation $P_j \tilde{\rho}_j = C_j P_j \sum_{k=1}^N \tilde{\rho}_k$. If $C_j = 1$ this relation can be only fulfilled when $P_j \tilde{\rho}_k = 0$ for any k with $k \neq j$ and when therefore also $P_j P_k = 0$ which follows from representing the positive operator $\tilde{\rho}_k$ by Eq. (2.5). This means that the projectors onto the eigenspaces belonging to the largest eigenvalues of $\tilde{\rho}_j$ and $\tilde{\rho}_k$, respectively, have to be mutually orthogonal. Equation (3.18) therefore necessarily requires that for $k, j = 1, \dots, N$

$$P_k P_j = \delta_{kj}, \quad \Lambda_k \rho_j = \Pi_k \rho_j = 0 \quad \text{for } k \neq j, \quad (3.19)$$

where the second relation follows from the first with the help of Eqs. (2.9) and (2.10). The last equality represents the condition for unambiguous discrimination. Conversely, Eq. (2.3) shows immediately that Eq. (3.18) follows from Eq. (3.19).

We emphasize that when the maximum confidence is equal to unity for some of the states and smaller for the rest of them, the strategies of maximum-confidence discrimination and of unambiguous discrimination are different. In unambiguous discrimination the detection operator Π_j is zero for a state where $C_j < 1$ since this state cannot be unambiguously discriminated and therefore always yields an inconclusive result. Examples to elucidate this difference for $N = 2$ are presented in [21].

Let us specialize the previous considerations about a standard form of the optimization problem to the case where Eq. (3.18) holds. From Eqs. (3.19) and (2.5) it follows that $P = \sum_{j=1}^N P_j$ projects onto a subspace of dimension $d' = \sum_{j=1}^N m_j$. If $d' = d$, P is equal to the identity operator I_d which means that also $\Lambda = I_d$ since P_j is the support of Λ_j . Using Eq. (2.5) with $C_j = 1$ in order to calculate $\sum_{j=1}^N \tilde{\rho}_j$ and comparing the result with the resolution of the identity in Eq. (2.4) it follows that $\tilde{\rho}_j = P_j$ ($j = 1, \dots, N$). This implies that for $\Lambda = I_d$ there does not exist a common subspace for the supports of any two of the operators $\tilde{\rho}_j$ and hence also no common subspace for any two of the given density operators ρ_j . Hence the reduction of the optimization problem to the standard form, characterized by Eq. (3.17), is equivalent to the elimination of common subspaces between the given density operators. Our treatment provides a recipe how to perform this elimination, thus extending the corresponding reduction theorem established for the optimum unambiguous discrimination of two mixed states [7] to an arbitrary number N of mixed states.

E. Maximum-confidence discrimination for $N = 2$

The case $N = 2$ is exceptional since in this case the relation $P_1 P_2 = 0$, which according to Eq. (3.19) holds when $C_1 = C_2 = 1$, that is for unambiguous discrimination, remains valid for any maximum-confidence measurement, with arbitrary values C_1 and C_2 . This is due to the fact that the transformed density operators $\tilde{\rho}_1$ and $\tilde{\rho}_2 = I_d - \tilde{\rho}_1$ in Eq. (2.5) have the same system of

eigenstates. The eigenstates of $\tilde{\rho}_1$ belonging to its largest eigenvalue are associated with the smallest eigenvalue of $\tilde{\rho}_2$, and vice versa [21], and the projectors P_1 and P_2 are therefore orthogonal.

When $P_1 + P_2 = I_d$, that is when the optimization problem has been reduced to the standard form characterized by Eq. (3.17), we obtain from Eq. (2.5)

$$\tilde{\rho}_1 = C_1 P_1 + (1 - C_2) P_2, \quad \tilde{\rho}_2 = C_2 P_2 + (1 - C_1) P_1. \quad (3.20)$$

Let us introduce the operators $\sigma_j = \rho^{1/2} P_j \rho^{1/2}$ ($j = 1, 2$) with $\sigma_1 + \sigma_2 = \rho$. Making use of Eq. (2.10), we find that

$$\sigma_1 \Lambda_2 = \sigma_2 \Lambda_1 = 0, \quad \Lambda_j \rho \Lambda_j = \Lambda_j \sigma_j \Lambda_j = \Lambda_j \sigma'_j \Lambda_j \text{Tr} \sigma_j, \quad (3.21)$$

where the operators $\sigma'_j = (\text{Tr} \sigma_j)^{-1} \sigma_j$ can be interpreted as density operators. The first equation implies that $\sigma'_1 \Pi_2 = \sigma'_2 \Pi_1 = 0$, due to Eq. (2.9). Hence after substituting the second equation into the optimality conditions, Eqs. (3.9) and (3.10), we find that the conditions for the optimized maximum-confidence discrimination between ρ_1 and ρ_2 are equivalent to the conditions for the optimum unambiguous discrimination between σ'_1 and σ'_2 , occurring with the prior probabilities $\text{Tr} \sigma_1$ and $\text{Tr} \sigma_2$, respectively. The results obtained from studying the latter problem [6–13] are therefore directly applicable to the former. In explicit terms, the operators $\sigma_j = \rho^{1/2} P_j \rho^{1/2}$ read

$$\sigma_1 = \frac{\eta_1 \rho_1 C_2 - \eta_2 \rho_2 (1 - C_1)}{C_1 + C_2 - 1}, \quad \sigma_2 = \frac{\eta_2 \rho_2 C_1 - \eta_1 \rho_1 (1 - C_2)}{C_1 + C_2 - 1}, \quad (3.22)$$

as follows from Eqs. (3.20) and (2.4).

IV. APPLICATION TO SYMMETRIC STATES

A. Properties of the optimum measurement

In the following we assume that the N states occur with equal prior probability $1/N$, and that they are cyclically symmetric in the sense that neighboring states arise from each other by the same unitary transformation. The density operators are given as

$$\rho_{j+1} = V^j \rho_1 V^{\dagger j} \quad \text{with} \quad V^\dagger V = V^N = I_d \quad (4.1)$$

for $j = 0, \dots, N-1$, where without lack of generality ρ_1 has been taken as the reference operator. By renumbering the states it can be easily seen that [26]

$$\rho = \frac{1}{N} \sum_{j=1}^N \rho_j = V \rho V^\dagger, \quad [V, \rho] = 0, \quad (4.2)$$

where the second equation follows from the first. Since the Hermitian operator ρ and the unitary operator V commute the two operators have the same eigenbasis in \mathcal{H}_d . Upon denoting their eigenstates by $\{|r_l\rangle\}$ with $l =$

$1, \dots, d$, in correspondence with Eq. (2.1), we get from Eq. (4.1) the spectral representation

$$V = \sum_{l=1}^d v_l |r_l\rangle \langle r_l| \quad \text{with} \quad \sum_{j=1}^N (v_l v_{l'}^*)^j = N \delta_{ll'} \quad (4.3)$$

and with $|v_l|^2 = v_l^N = 1$, yielding

$$\langle r_l | \rho_j | r_l \rangle = \langle r_l | \rho | r_l \rangle \equiv r_l \quad (j = 1, \dots, N). \quad (4.4)$$

This shows that in the eigenbasis of the symmetry operator the symmetric density operators have the same diagonal elements. The second equality in Eq. (4.3) follows from expanding the density operator ρ_1 in terms of the eigenbasis of ρ or V , respectively, arriving at

$$\rho = \frac{1}{N} \sum_{j=1}^N V^j \rho_1 V^{\dagger j} = \sum_{l, l'=1}^d \langle r_l | \rho_1 | r_{l'} \rangle \sum_{j=1}^N \frac{(v_l v_{l'}^*)^j}{N} |r_l\rangle \langle r_{l'}|. \quad (4.5)$$

Taking Eq. (4.4) into account, the desired equality is immediately obvious by comparison with Eq. (2.1).

After these general considerations about symmetric states we now focus on their maximum-confidence discrimination. Using $[V, \rho] = 0$ we find from Eqs. (2.4) and (4.1) that for symmetric states the transformed density operators obey the equation

$$\tilde{\rho}_{j+1} = \frac{1}{N} \rho^{-1/2} \rho_{j+1} \rho^{-1/2} = V^j \tilde{\rho}_1 V^{\dagger j} \quad (4.6)$$

with $j = 0, \dots, N-1$. Clearly, the eigenvalue spectrum is the same for each of the states $\tilde{\rho}_j$. As a consequence, also the maximum confidence for each of the outcomes, being equal to the largest eigenvalues of the transformed operators $\tilde{\rho}_j$, is the same,

$$C_1 = \dots = C_N \equiv C. \quad (4.7)$$

We mention at this point that whenever in a discrimination measurement the confidence has the same value C for each of the outcomes, then in this measurement the relation $\eta_j \text{Tr}(\rho_j \Pi_j) = C \text{Tr}(\rho \Pi_j)$ is fulfilled for $j = 1, \dots, N$, as becomes obvious from the definition of the confidence in the text before Eq. (2.3). Summation over all states on both sides of the latter equation yields

$$P_{\text{corr}} = \sum_{j=1}^N \eta_j \text{Tr}(\rho_j \Pi_j) = C \sum_{j=1}^N \text{Tr}(\rho \Pi_j) = C(1 - Q) \quad (4.8)$$

where P_{corr} is the overall probability of getting a correct result.

Due to Eq. (4.6) the projectors P_j onto the eigenspaces belonging to the largest eigenvalue of the operators $\tilde{\rho}_j$, see Eq. (2.5), obey the same symmetry as the given density operators. Since ρ and V commute, this symmetry is conveyed to the projectors Λ_j defined in Eq. (2.8) which specify the supports of the detection operators Π_j . We thus get

$$P_{j+1} = V^j P_1 V^{\dagger j}, \quad \Lambda_{j+1} = V^j \Lambda_1 V^{\dagger j}. \quad (4.9)$$

The optimized maximum-confidence measurement, minimizing the failure probability, is determined by the optimal detection operators satisfying Eqs. (3.5) and (3.7). Let us assume that Π_1 is an element of the set of optimal detection operators and consider the operators

$$\Pi_{j+1} = V^j \Pi_1 V^{\dagger j} \quad (4.10)$$

with $j = 0, \dots, N-1$. By the same arguments that led to the derivation of the commutation relation in Eq. (4.2) we find that the operator $\sum_{j=1}^N \Pi_j = I - \Pi_0$ commutes with V . This implies

$$[\Pi_0, V] = [\Pi_0, \rho] = 0, \quad [Z, V] = [Z, \rho] = 0, \quad (4.11)$$

where the second relation follows from the first when we take into account that $[Z, \Pi_0] = 0$, due to Eq. (3.7) and the hermiticity of the operators Z and Π_0 . Using Eqs. (4.9) – (4.11) it follows that for $j = 0, \dots, N-1$

$$\Lambda_{j+1}(Z - \rho)\Lambda_{j+1} = V^j \Lambda_1(Z - \rho)\Lambda_1 V^{\dagger j}. \quad (4.12)$$

Because of Eq. (2.9) this means that if Π_1 fulfills the optimality conditions, then these conditions are fulfilled by any of the operators Π_j defined in Eq. (4.10). Hence the detection operators for the optimum measurement can always be chosen in the form of Eq. (4.10), that is in a form where they have the same symmetry as the density operators. The optimality conditions therefore reduce to Eq. (4.10) together with

$$\Lambda_1(Z - \rho)\Lambda_1 = \Lambda_1\left(Z - \frac{\rho_1}{N C}\right)\Lambda_1 \geq 0, \quad Z \geq 0, \quad (4.13)$$

$$\Lambda_1(Z - \rho)\Pi_1 = \Lambda_1\left(Z - \frac{\rho_1}{N C}\right)\Pi_1 = 0, \quad Z\Pi_0 = 0, \quad (4.14)$$

where we used Eqs. (3.9) and (3.10). We mention that for the case of optimum unambiguous discrimination of symmetric states the corresponding symmetry property of the optimum detection operators was derived by Chefles and Barnett [27] for pure states and by Eldar *et al.* [8] for mixed states. From Eqs. (4.10) and (4.11) it follows that in the optimum measurement the probability R of conclusive results, defined in Eq. (3.1), can be written as

$$R = 1 - Q = N \text{Tr}(\rho \Pi_1) = \sum_{l=1}^d r_l N \langle r_l | \Pi_1 | r_l \rangle, \quad (4.15)$$

where the spectral representation of ρ has been used. Applying the properties of the symmetry operator V , given in Eq. (4.3), we arrive at the failure operator

$$\Pi_0 = I - \sum_{j=1}^N V^j \Pi_1 V^{\dagger j} = \sum_{l=1}^d (1 - N \langle r_l | \Pi_1 | r_l \rangle) |r_l\rangle \langle r_l|. \quad (4.16)$$

The optimum detection operator Π_1 maximizes R under the constraint that $\Pi_0 \geq 0$ and obeys the condition $\Pi_1 = \Lambda_1 \Pi_1 \Lambda_1$, which guarantees maximum-confidence discrimination.

According to Eq. (2.10) the projector Λ_1 is determined by the projector P_1 which in turn, together with the maximum confidence C , follows from the spectral decomposition of $\tilde{\rho}_1 = N^{-1} \rho^{-1/2} \rho_1 \rho^{-1/2}$. After expanding $\tilde{\rho}_1$ in terms of the eigenstates of ρ or V , respectively, Eq. (2.5) takes the form

$$\tilde{\rho}_1 = \sum_{l, l'=1}^d \frac{\langle r_l | \rho_1 | r_{l'} \rangle}{N \sqrt{r_l r_{l'}}} |r_l\rangle \langle r_{l'}| = C P_1 + \sum_{k=m+1}^d \nu_k^{(1)} |\nu_k^{(1)}\rangle \langle \nu_k^{(1)}| \quad (4.17)$$

with $P_1 = \sum_{k=1}^m |\nu_k^{(1)}\rangle \langle \nu_k^{(1)}|$. Here $m = \text{rank } P_1$ is the degeneracy of the largest eigenvalue, C . Clearly, the rank of P_1 depends on the matrix elements $\langle r_l | \rho_1 | r_{l'} \rangle$, that is on the representation of ρ_1 in the eigenbasis of the symmetry operator.

B. General solution for one-dimensional detection operators

In the rest of the paper we suppose that the largest eigenvalue of the transformed density operator $\tilde{\rho}_1$ is non-degenerate, that is $m = 1$ in Eq. (4.17). Provided that the spectral representations of ρ and $\tilde{\rho}_1$ are known, in this simple case the optimization problem posed in Eq. (3.2) can be readily solved in a direct calculation, without resorting to the general optimality conditions. For convenience we drop the superscript which indicates the number of the state, that is we use the notation $|\nu_1^{(1)}\rangle = |\nu_1\rangle$. With the help of Eq. (2.10) we then get

$$P_1 = |\nu_1\rangle \langle \nu_1|, \quad \Lambda_1 = \rho^{-1/2} \frac{|\nu_1\rangle \langle \nu_1|}{\langle \nu_1 | \rho^{-1} | \nu_1 \rangle} \rho^{-1/2}, \quad (4.18)$$

where $|\nu_1\rangle$ is the eigenstate of $\tilde{\rho}_1$ belonging to its largest eigenvalue. The requirement $\Pi_1 = \Lambda_1 \Pi_1 \Lambda_1$ guaranteeing maximum-confidence discrimination leads to the Ansatz

$$\Pi_1 = \alpha \rho^{-1/2} |\nu_1\rangle \langle \nu_1| \rho^{-1/2} \quad (4.19)$$

which because of Eqs. (4.15) and (4.16) yields

$$R = N\alpha, \quad \Pi_0 = \sum_{l=1}^d \left(1 - N\alpha \frac{|\langle r_l | \nu_1 \rangle|^2}{r_l}\right) |r_l\rangle \langle r_l|. \quad (4.20)$$

In order to maximize R we have to find the largest admissible value of α which is consistent with the constraint $\Pi_0 \geq 0$. Clearly, this constraint requires that $\alpha N |\langle r_l | \nu_1 \rangle|^2 \leq r_l$ for each value of l . The largest possible value of α and the resulting minimum failure probability $Q = 1 - R$ are therefore

$$\alpha_{\text{opt}} = \frac{1}{N} \text{Min}_l \left\{ \frac{r_l}{|\langle r_l | \nu_1 \rangle|^2} \right\}, \quad Q_{\text{min}} = 1 - N\alpha_{\text{opt}}, \quad (4.21)$$

where the minimum is taken with respect to the different values of l .

In general, there can exist l_0 different values of l for which the fraction $r_l/|\langle r_l|\nu_1\rangle|^2$ takes the same minimal value, where $1 \leq l_0 \leq d$. Let us number the eigenstates $|r_l\rangle$ in such a way that the condition $r_l = \alpha_{opt}N|\langle r_l|\nu_1\rangle|^2$ is fulfilled for $l = 1, \dots, l_0$, yielding the failure operator

$$\Pi_0 = \sum_{l=l_0+1}^d \left(1 - N\alpha_{opt} \frac{|\langle r_l|\nu_1\rangle|^2}{r_l}\right) |r_l\rangle\langle r_l| \quad (4.22)$$

with $\text{rank } \Pi_0 = d - l_0$. Clearly, $\text{rank } \Pi_0 \leq d - 1$, in accordance with Eq. (3.14). For $l_0 = d$ we get $\Pi_0 = 0$ which means that in this special case inconclusive results do not occur in the optimized maximum-confidence measurement. This case arises when $r_l = |\langle r_l|\nu_1\rangle|^2$ for each value of l , that is when $\rho = |\nu_1\rangle\langle\nu_1|$. An example will be treated in Sec. IV.C.

It is easy to verify that the solution fulfills the necessary and sufficient optimality conditions. For this purpose we introduce the operator

$$Z = \frac{N\alpha_{opt}}{l_0} \sum_{l=1}^{l_0} |r_l\rangle\langle r_l| = \frac{1}{l_0} \sum_{l=1}^{l_0} \frac{r_l}{|\langle r_l|\nu_1\rangle|^2} |r_l\rangle\langle r_l| \quad (4.23)$$

which is obviously positive and orthogonal to Π_0 , that is $Z \geq 0$ and $Z\Pi_0 = 0$. Moreover, we obtain

$$\langle \nu_1 | \rho^{-1/2} (Z - \rho) \rho^{-1/2} | \nu_1 \rangle = 0 \quad (4.24)$$

which because of Eq. (4.18) means that $\Lambda_1(Z - \rho)\Lambda_1 = 0$ and therefore also $\Lambda_1(Z - \rho)\Pi_1 = 0$. Hence Eqs. (4.13) and (4.14) are satisfied.

C. Examples

In our examples we consider the discrimination of N equiprobable symmetric mixed states of the form

$$\rho_j = p |\psi_j\rangle\langle\psi_j| + \frac{1-p}{d} I_d \quad (j = 1, \dots, N), \quad (4.25)$$

where $|\psi_j\rangle$ is normalized to unity and where the parameter p with $0 \leq p \leq 1$ is related to the purity of the given states. The symmetry of the set of mixed states ρ_j requires that

$$|\psi_{j+1}\rangle = V^j |\psi_1\rangle \quad \text{with} \quad V^\dagger V = V^N = I_d. \quad (4.26)$$

If $N < d$ the N pure states $|\psi_j\rangle$ span a Hilbert space of dimension d' with $d' < d$ and the optimization problem can be reduced to state discrimination within the d' -dimensional Hilbert space. Therefore without lack of generality we assume that the N states $|\psi_j\rangle$ span the full d -dimensional Hilbert space. This means that $N \geq d$ and that the expansion of $|\psi_1\rangle$ with respect to the eigenbasis of V reads

$$|\psi_1\rangle = \sum_{l=1}^d c_l |r_l\rangle \quad \text{with} \quad c_l \neq 0 \quad \text{for} \quad l = 1, \dots, d. \quad (4.27)$$

Making use of Eqs. (4.1) and (4.3) we arrive at

$$\rho = \sum_{j=1}^N \frac{\rho_j}{N} = \sum_{l=1}^d r_l |r_l\rangle\langle r_l| \quad \text{with} \quad r_l = p |c_l|^2 + \frac{1-p}{d}. \quad (4.28)$$

The general expression in Eq. (4.17) then takes the form

$$\tilde{\rho}_1 = \rho^{-1/2} \frac{\rho_1}{N} \rho^{-1/2} = \frac{p}{N} \sum_{\substack{l,l' \\ (l \neq l')}}^d \frac{c_l c_{l'}^*}{\sqrt{r_l r_{l'}}} |r_l\rangle\langle r_{l'}| + \frac{I_d}{N}. \quad (4.29)$$

In order to determine the optimum measurement, we need to find the spectral decomposition of $\tilde{\rho}_1$. In the following we restrict ourselves to simple cases where this task can be solved analytically.

1. N symmetric pure states in a d -dimensional joint Hilbert space ($N \geq d$)

First we treat the case that in Eq. (4.25) $p = 1$ which means that the states to be discriminated are the N equiprobable symmetric pure qudit states $|\psi_1\rangle \dots |\psi_N\rangle$. Clearly, when $N > d$ the states are linearly dependent. After substituting $p = 1$ and $\sqrt{r_l} = |c_l|$, the operator $\tilde{\rho}_1$ in Eq. (4.29) takes the form

$$\tilde{\rho}_1 = \frac{d}{N} |\nu_1\rangle\langle\nu_1| \quad \text{with} \quad |\nu_1\rangle = \frac{1}{\sqrt{d}} \sum_{l=1}^d \frac{c_l}{|c_l|} |r_l\rangle = \frac{\rho^{-1/2}}{\sqrt{d}} |\psi_1\rangle. \quad (4.30)$$

From Eq. (2.5) we obtain the maximum confidence C , and Eq. (4.21) with $r_l = |c_l|^2$ yields α_{opt} and the minimum failure probability Q_{min} . We thus arrive at

$$C = \frac{d}{N}, \quad Q_{min} = 1 - d \min_l \{|c_l|^2\}. \quad (4.31)$$

Taking Eqs. (4.19), (4.30) and (4.10) into account, the optimum detection operators can be represented as

$$\Pi_j = \frac{\min_l \{|c_l|^2\}}{N} \rho^{-1} |\psi_j\rangle\langle\psi_j| \rho^{-1} \quad (j = 1, \dots, N), \quad (4.32)$$

where $\rho = \frac{1}{N} \sum_{j=1}^N |\psi_j\rangle\langle\psi_j|$. The maximum confidence is solely determined by the dimension d of the Hilbert space spanned by the states and does not depend on their special form. For $N = 3$ and $d = 2$ the expression for the maximum confidence is in accordance with the result obtained in [18].

When $N = d$, that is when the pure states are linearly independent and $C = 1$, the expression for Q_{min} in Eq. (4.31) coincides with the minimum failure probability necessary for the unambiguous discrimination of N linearly independent symmetric pure states, derived by Chefles and Barnett [27]. Indeed, since for these states the eigenvalues of the symmetry operator can be represented as $v_l = \exp(2\pi i \frac{l}{N})$ [27] we find from Eqs. (4.26)

and (4.28) with $r_l = |c_l|^2$ that

$$\langle \psi_j | \rho^{-1} | \psi_k \rangle = \sum_{l=1}^d v_l^{*j} \frac{|c_l|^2}{r_l} v_l^k = \sum_{l=1}^N \left(e^{2\pi i \frac{j-k}{N}} \right)^l = N \delta_{jk}, \quad (4.33)$$

yielding due to Eq. (4.32) $\Pi_j | \psi_k \rangle = 0$ for $j \neq k$, which is the condition for unambiguous discrimination. When $|c_l| = 1/\sqrt{d}$ for $d = N$, the N given states are mutually orthogonal and $\rho = I_d/d$. The optimum measurement is then projective with $\sum_{j=1}^N \Pi_j = I_d$ following from Eq. (4.32) since $\rho^{-1} = dI_d$.

It is interesting to compare the maximum confidence C with the confidence C_{ME} that is achieved in minimum-error discrimination, where inconclusive results do not occur and the probability of correct results, P_{corr} , is maximal. For symmetric pure states the minimum-error measurement is known to be the square-root measurement described by the detection operators [26]

$$\Pi_j^{ME} = \frac{1}{N} \rho^{-1/2} | \psi_j \rangle \langle \psi_j | \rho^{-1/2} = \frac{d}{N} V^j | \nu_1 \rangle \langle \nu_1 | V^{\dagger j} \quad (4.34)$$

where for the second equality sign we applied Eqs. (4.26) and (4.30). From Eq. (4.8) with $Q = 0$ we obtain $P_{corr}^{ME} = C_{ME}$. Using the operators Π_j^{ME} and $\eta_j = 1/N$ we arrive at

$$C_{ME} = \frac{d}{N} |\langle \psi_1 | \nu_1 \rangle|^2 = C |\langle \psi_1 | \nu_1 \rangle|^2 = \frac{C}{d} \left(\sum_{l=1}^d |c_l| \right)^2 \quad (4.35)$$

which shows that $C_{ME} \leq C$ as expected. Obviously the gain in confidence achieved by admitting inconclusive results and performing a maximum-confidence measurement depends on the expansion coefficients c_l of the given pure states with respect to the eigenbasis of the symmetry operator. Equality only holds when $|c_l| = 1/\sqrt{d}$ for each value of l . In this case Eq. (4.31) shows that there are no inconclusive results in the maximum-confidence measurement. In addition, it follows that the detection operators in Eqs. (4.32) and (4.34) are identical, that is the minimum-error measurement and the optimized maximum-confidence measurement coincide.

2. N symmetric mixed qubit states

In our second example we consider the discrimination of N symmetric mixed states of rank 2 in a two-dimensional joint Hilbert space. Since an arbitrary mixed qubit state can be always written in the form of Eq. (4.25) with $d = 2$ and with a certain value of the parameter p ($p \neq 0$), the maximum confidence C corresponds to the largest eigenvalue of the operator $\tilde{\rho}_1$ in Eq. (4.29) with $d = 2$. Upon determining the spectral representation of this operator, using the expansion $|\psi_1\rangle = c_1|r_1\rangle + c_2|r_2\rangle$, where $|r_1\rangle$ and $|r_2\rangle$ are the eigenstates of the symmetry operator V , we obtain both the

largest eigenvalue, equal to the maximum confidence,

$$C = \frac{1}{N} \left[1 + \frac{p |c_1 c_2|}{\sqrt{(p|c_1|^2 + \frac{1-p}{2})(p|c_2|^2 + \frac{1-p}{2})}} \right] \quad (4.36)$$

and the corresponding eigenstate $|\nu_1\rangle = \frac{1}{\sqrt{2}}(|r_1\rangle + |r_2\rangle)$. Equation (4.21) yields

$$\alpha_{opt} = \frac{1}{N} (1 - p + 2p \min\{|c_1|^2, |c_2|^2\}) \quad (4.37)$$

which in turn determines the optimum detection operator Π_1 , see Eq. (4.19), as well as the minimum failure probability necessary for maximum-confidence discrimination,

$$Q_{min} = p (1 - 2 \min\{|c_1|^2, |c_2|^2\}) \quad (4.38)$$

Introducing

$$|c_1| = \cos \frac{\gamma}{2}, \quad |c_2| = \sin \frac{\gamma}{2} \quad \text{with } 0 \leq \gamma \leq \pi/2, \quad (4.39)$$

that is, $\min\{|c_1|^2, |c_2|^2\} = \sin^2 \frac{\gamma}{2}$, the above equations can be rewritten as

$$C = \frac{1}{N} \left(1 + \frac{p \sin \gamma}{\sqrt{1 - p^2 \cos^2 \gamma}} \right), \quad Q_{min} = p \cos \gamma. \quad (4.40)$$

Clearly, the maximum value of the confidence decreases with growing number of states while the minimum failure probability necessary for maximum-confidence discrimination stays constant.

We note that two arbitrary mixed qubit states in the same Hilbert space and with the same purity always belong to the class of symmetric states. They can be represented by using Eqs. (4.25) and (4.26) with $N = d = 2$ and with $|\psi_{1,2}\rangle = \cos^2 \frac{\gamma}{2} |0\rangle \pm e^{i\phi} \sin^2 \frac{\gamma}{2} |1\rangle$ where the states $|0\rangle$ and $|1\rangle$ are orthonormal basis states and the symmetry operator is given as $V = |0\rangle\langle 0| - |1\rangle\langle 1|$. With $|\langle \psi_1 | \psi_2 \rangle| = \cos \gamma$ Eq. (4.40) corresponds to our earlier result for the maximum confidence discrimination of two equally probable mixed qubit states having the same purity [21].

3. N special symmetric mixed states in a d -dimensional joint Hilbert space ($N \geq d$)

Our last example refers to the case that in Eq. (4.25) $p \neq 0$ and the dimension d of the joint Hilbert space is arbitrary. However, we assume that the N states $|\psi_j\rangle$ are of the special kind where the modulus of all expansion coefficients is the same, that is, where Eqs. (4.27) – (4.29) take the form

$$|\psi_1\rangle = \sum_{i=0}^d \frac{|r_i\rangle}{\sqrt{d}}, \quad \rho = \frac{I_d}{d}, \quad \tilde{\rho}_1 = \frac{I_d(1-p)}{N} + \frac{pd}{N} |\psi_1\rangle \langle \psi_1|. \quad (4.41)$$

The largest eigenvalue of $\tilde{\rho}_1$, belonging to the eigenstate $|\nu_1\rangle = |\psi_1\rangle$ and determining the maximum confidence, C , can be immediately read out. Using Eqs. (4.19) – (4.21) we find that

$$C = \frac{1 + p(d-1)}{N}, \quad \alpha_{opt} = \frac{1}{N}, \quad Q_{min} = 0. \quad (4.42)$$

The optimum detection operators are

$$\Pi_j = \frac{d}{N} |\psi_j\rangle\langle\psi_j| \quad (j = 1, \dots, N) \quad (4.43)$$

where Eq. (4.32) with $\rho = I_d/d$ has been taken into account. Clearly, in this special case the optimized maximum-confidence measurement for discriminating the given mixed states does not require inconclusive results, that is, $\Pi_0 = 0$. Comparison with Eq. (4.34) shows that the measurement is equal to the minimum-error measurement for discriminating the underlying pure states $|\psi_j\rangle$.

V. SUMMARY AND CONCLUDING REMARKS

To summarize, in this paper we derived necessary and sufficient optimality conditions for a measurement which discriminates N mixed quantum states with maximum confidence for each conclusive outcome, thereby keeping the overall probability of inconclusive outcomes as small as possible. These conditions are given by Eqs. (3.5) and (3.7) together with Eqs. (2.9) and (2.10). They generalize earlier optimality conditions [8] which refer to the special case of optimum unambiguous discrimination. We derived general properties of the optimum measurement and applied the optimality conditions to equiprobable symmetric states. For these states we presented analytical solutions of the optimization problem for examples where the detection operators describing the maximum-confidence discrimination are one-dimensional.

When higher-rank detection operators are involved, the general problem of minimizing the failure probability gets similarly complicated for maximum-confidence discrimination as it is for unambiguous discrimination. As shown in Sec. III E, when $N = 2$ both problems are mathematically equivalent. It has been found that already for the simplest general case of higher-rank detection operators in unambiguous discrimination, that is for the discrimination of two density operators of rank 2 in a four-dimensional Hilbert space, the optimization problem can in general lead to polynomial equations of higher degree [13]. For $N \geq 3$ a general solution is not known for the unambiguous discrimination of mixed states and only bounds have been derived [14, 15]. However, analytical solutions with higher-rank detection operators can be constructed in special cases where the given density operators allow us to separate the optimization problem into independent optimization problems in mutually orthogonal subspaces of the Hilbert space and where the projections of the detection operators onto these subspaces

are one-dimensional. This method has been applied for the optimum unambiguous discrimination with $N = 2$, see, e. g., [9–11], and also for a case of equiprobable symmetric mixed states with arbitrary N [28], as well as for the optimized maximum-confidence discrimination of two mixed states, where an example was given in [21].

We still remark that other state-discrimination strategies have been introduced where the overall probability of getting a correct result, P_{corr} , is maximized under the constraint that either the failure probability [29–31] or the error probability [32–34] has a certain fixed value. Maximum-confidence discrimination is related to the former of these, as discussed already in [21] and [35]. In fact, when for the given states the maximum achievable confidence is the same for each individual outcome, that is, when $C_j = C$ for $j = 1, \dots, N$, the maximum-confidence measurement where the failure probability takes its minimum, Q_{min} , coincides with the measurement which maximizes P_{corr} under the condition that Q is fixed at the value Q_{min} . This is due to the fact that the latter measurement also maximizes the ratio $P_{corr}/(1-Q)$ at the fixed value of Q and that according to Eq. (4.8) this ratio is equal to the confidence. Hence in this case maximizing P_{corr} at a fixed value Q with $0 \leq Q \leq Q_{min}$ corresponds to interpolating between minimum-error discrimination for $Q = 0$, and optimized maximum-confidence discrimination for $Q = Q_{min}$, or optimum unambiguous discrimination, respectively, if $C = 1$.

On the other hand, when the maximum confidence differs for the individual outcomes, it follows that the maximum of $P_{corr}/(1-Q)$ at a fixed value of Q is equal to $\max_j \{C_j\}$, and that it is obtained in a modified maximum-confidence measurement where all states j with $C_j < \max_j \{C_j\}$ yield an inconclusive result [21]. The failure probability resulting from a measurement of this kind is not necessarily the smallest one that can be reached in maximum-confidence discrimination. For two mixed qubit states with arbitrary values of C_1 and C_2 , defined in the same Hilbert space, maximum-confidence discrimination with minimum failure probability has been studied in our earlier paper [21]. In order to obtain solutions for discriminating more than two states, the optimality conditions derived in this paper can be applied.

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Appendix

We want to show that Eq. (3.5) is not only sufficient, but also necessary for Eq. (3.4) to hold. In analogy to a recent treatment of minimum-error discrimination [25], we perform the proof by demonstrating an example

where a single negative eigenvalue of one of the operators in Eq. (3.5) leads to a violation of Eq. (3.4).

Let us first assume that a particular one of the operators $\Lambda_j(Z - \rho)\Lambda_j$, say the one with $j = N$, has a negative eigenvalue $-\mu$, resulting in the eigenvalue equation

$$\Lambda_N(Z - \rho)\Lambda_N|\mu\rangle = -\mu|\mu\rangle \quad \text{with } \mu > 0. \quad (\text{A.1})$$

Now we suppose that the detection operators Π_0, \dots, Π_N are optimal which means that they obey the equalities in Eq. (3.7). In analogy to [25] we define another set of operators, given as

$$\Pi'_j = (I_d - \epsilon|\mu\rangle\langle\mu|) \Pi_j (I_d - \epsilon|\mu\rangle\langle\mu|) \quad (\text{A.2})$$

for $j = 0, 1, \dots, N - 1$ and

$$\Pi'_N = (I_d - \epsilon|\mu\rangle\langle\mu|)\Pi_N(I_d - \epsilon|\mu\rangle\langle\mu|) + \epsilon(2 - \epsilon)|\mu\rangle\langle\mu| \quad (\text{A.3})$$

where $0 \leq \epsilon \ll 1$. It can be easily checked by a straight-forward calculation that the primed operators fulfill the conditions for completeness and positivity expressed by Eq. (2.2) and are thus a valid set of detection operators, yielding the discrimination probability $R' = \sum_{j=1}^N \text{Tr}(\rho\Lambda_j\Pi'_j\Lambda_j)$. Using Eqs. (A.1) – (A.3) as well as the completeness relation $\sum_{j=0}^N \Pi'_j = I_d$ we obtain

$$\begin{aligned} \text{Tr}Z - R' &= \text{Tr}(Z\Pi'_0) + \sum_{j=1}^N \text{Tr}[\Lambda_j(Z - \rho)\Lambda_j\Pi'_j] \quad (\text{A.4}) \\ &= \epsilon(2 - \epsilon)\text{Tr}[\Lambda_N(Z - \rho)\Lambda_N|\mu\rangle\langle\mu|] = -2\epsilon\mu + O(\epsilon^2) \end{aligned}$$

where for the second equality sign Eq. (3.7) has been taken into account. Hence the negativity of $\Lambda_N(Z - \rho)\Lambda_N$ implies that $\text{Tr}Z - R' < 0$, in contrast to Eq. (3.4). Thus we have shown that the positivity of all operators $\Lambda_j(Z - \rho)\Lambda_j$ is a necessary condition for Eq. (3.4).

With respect to the positivity of Z the proof proceeds in a completely analogous way. We now assume that

$$Z|\mu\rangle = -\mu|\mu\rangle \quad \text{with } \mu > 0 \quad (\text{A.5})$$

and we suppose that the primed detection operators are determined by Eq. (A.2) for $j = 1, \dots, N$ while Π'_0 is given by Eq. (A.3) with N replaced by 0. Applying Eq. (3.7) we then again find that

$$\text{Tr}Z - R' = -2\epsilon\mu + O(\epsilon^2) < 0, \quad (\text{A.6})$$

which means that the negativity of Z leads to a violation of Eq. (3.4), or, in other words, that the positivity of Z is a necessary condition for Eq. (3.4).

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- [1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
 - [2] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).
 - [3] D. Dieks, Phys. Lett. A **126**, 303 (1988).
 - [4] A. Peres, Phys. Lett. A **128**, 19 (1988).
 - [5] G. Jaeger and A. Shimony, Phys. Lett. A **197**, 83 (1995).
 - [6] T. Rudolph, R. W. Spekkens, and P. S. Turner, Phys. Rev. A **68**, 010301(R) (2003).
 - [7] Ph. Raynal, N. Lütkenhaus, and S. van Enk, Phys. Rev. A **68**, 022308 (2003).
 - [8] Y. C. Eldar, M. Stojnic, and B. Hassibi, Phys. Rev. A **69**, 062318 (2004).
 - [9] U. Herzog and J. A. Bergou, Phys. Rev. A **71**, 050301(R) (2005).
 - [10] J. A. Bergou, E. Feldman, and M. Hillery, Phys. Rev. A **73**, 032107 (2006).
 - [11] U. Herzog, Phys. Rev. A **75**, 052309 (2007).
 - [12] Ph. Raynal and N. Lütkenhaus, Phys. Rev. A **76**, 052322 (2007).
 - [13] M. Kleinmann, H. Kampermann, and D. Bruß, Phys. Rev. A **81**, 020304(R) (2010); J. Math. Phys. **51**, 032201 (2010).
 - [14] Y. Feng, R. Duan, and M. Ying, Phys. Rev. A **70**, 012308 (2004).
 - [15] C. Zhang, Y. Feng, M. Ying, Phys. Lett. A **353**, 300 (2006).
 - [16] A. Chefles, Phys. Lett. A **239**, 339 (1998).
 - [17] The support of a density operator is the Hilbert space spanned by its eigenvectors with nonzero eigenvalues. The rank is the dimension of the support.
 - [18] S. Croke, E. Andersson, S. M. Barnett, C. R. Gilson and J. Jeffers, Phys. Rev. Lett. **96**, 070401 (2006).
 - [19] S. Croke, E. Andersson and S. M. Barnett, Phys. Rev. A **77**, 012113 (2008).
 - [20] P. J. Mosley, S. Croke, I. A. Walmsley and S. M. Barnett, Phys. Rev. Lett. **97**, 193601 (2006).
 - [21] U. Herzog, Phys. Rev. A **79**, 032323 (2009).
 - [22] U. Herzog and O. Benson, J. Mod. Opt. **57**, 188 (2010).
 - [23] G. A. Steudle, S. Knauer, U. Herzog, E. Stock, V. Haisler, D. Bimberg, and O. Benson, Phys. Rev. A **83**, 050304(R) (2011).
 - [24] O. Jiménez, M. A. Solis-Prosser, A. Delgado, and L. Neves, Phys. Rev. A **84**, 062315 (2011).
 - [25] S. M. Barnett and S. Croke, J. Phys. A **42**, 062001 (2009).
 - [26] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, Int. J. Theor. Phys. **36**, 1269 (1997).
 - [27] A. Chefles and S. M. Barnett, Phys. Lett. A **250**, 223 (1998).
 - [28] U. Herzog and J. A. Bergou, Phys. Rev. A **78**, 032320

- (2008), Erratum: Phys. Rev. A **78**, 069902(E) (2008).
- [29] A. Chefles and S. M. Barnett, J. Mod. Opt. **45**, 1295 (1998).
- [30] J. Fiurášek and M. Ježek, Phys. Rev. A **67**, 012321 (2003).
- [31] Y. C. Eldar, Phys. Rev. A **67**, 042309 (2003).
- [32] M. A. P. Touzel, R. B. A. Adamson, and A. M. Steinberg, Phys. Rev. A **76**, 062314 (2007).
- [33] A. Hayashi, T. Hashimoto, and M. Horibe, Phys. Rev. A **78**, 012333 (2008).
- [34] H. Sugimoto, T. Hashimoto, M. Horibe, and A. Hayashi, Phys. Rev. A **80**, 052322 (2009).
- [35] S. M. Barnett and S. Croke, Adv. Opt. Photon. **1**, 238 (2009).